Groups of generalized flux transformations in the space of generalized connections

J. M. Velhinho

Departamento de Física, Universidade da Beira Interior
R. Marquês D'Ávila e Bolama, 6201-001 Covilhã, Portugal
jvelhi@ubi.pt

Abstract

We present a group of transformations in the space of generalized connections that contains the set of transformations generated by the flux variables of loop quantum gravity. This group is labelled by certain $SU(2)$-valued functions on the bundle of directions in the spatial manifold. A further generalization is obtained by considering functions that depend on germs of analytic curves, rather than just on directions.

1 Introduction and motivation

The space of generalized connections $\bar{\mathcal{A}}$ is a distributional extension of the space of smooth connections in a manifold [1, 2, 3, 4]. The space $\bar{\mathcal{A}}$ features in particular in the canonical approach to quantum gravity known as loop quantum gravity (LQG),[1] where it plays the role of a (kinematical) 'quantum configuration space'.

Following early ideas [1, 4], a category theory description of the space of generalized connections has been put forward [7, 8] and explored, aiming at a complete knowledge of the natural algebraic structures present in $\bar{\mathcal{A}}$. As an example of the interest of this type of formulations, we mention the natural emergence of a group of automorphisms (namely the automorphisms of the groupoid of paths) as a possible distributional extension of the group of spatial diffeomorphims in LQG. This extension, proposed in [8], was recently investigated in [9], where the relation with a combinatorial formulation of LQG is also explored.

A still unveiled structure in $\bar{\mathcal{A}}$ is the group of transformations generated by the LQG flux variables. Being of the momentum type, the flux variables correspond to infinitesimal 'translations' in the configuration space $\bar{\mathcal{A}}$. The one-parameter groups of transformations generated by each flux variable are well known [10, 11], and also both generalizations of these transformations [10, 12, 13]

---

[1]For reviews of LQG see e.g. [5, 6].

and categorical formulations [9] have been introduced. However, even including the generalizations, the set of transformations associated with the flux variables does not form a group. Therefore, in order to accommodate the composition of transformations generated by flux variables, one needs to consider a class of transformations in $\bar{\mathcal{A}}$ more general than those described so far.

Inspired by ideas presented in [6, 10], we introduce here a group of transformations in $\bar{\mathcal{A}}$ that includes all the transformations generated by the flux variables. Moreover, this group sits inside a larger group of transformations, of the same type, but depending on a larger amount of information. All these transformations leave invariant the so-called Ashtekar-Lewandowski measure, and so they are all unitarily implemented in the LQG (kinematical) Hilbert space.

As in the case of generalized (local) gauge transformations, the full group of generalized flux transformations in question can be seen as the action of a certain group of functions with values in the gauge group. In the present case, those functions depend in general on points and on germs of analytic curves in the spatial manifold. From a different perspective, this group can also be realized as a subset of $\bar{\mathcal{A}}$, and in this sense it includes the examples of distributional connections typically discussed in the literature.

In what follows, we will focus on the trivial bundle case and the analytic set-up. To avoid repeating the existing literature (see e.g. [6] and references therein), only the basic notions from the space of generalized connections are introduced. Also, only the main arguments of proofs are presented, obviating straightforward technical details.

2   Generalized flux transformations

Let Σ be a connected analytic manifold, G a compact, connected Lie group, and $P(\Sigma, G)$ the trivial principal G-bundle over Σ. ($G = SU(2)$ in LQG.)

Let us consider compactly supported, embedded, oriented analytic curves in Σ, i.e. maps $c : [0, 1] \to \Sigma$ whose image is contained in a compact set, which are analytic in the whole domain, and such that $c(]0, 1[)$ is an embedded submanifold. On the set of these curves one introduces an equivalence relation which identifies two curves if they differ only by an orientation preserving analytic reparametrization. Thus, two equivalent curves have the same image and the same orientation. The equivalence classes of such oriented analytic curves are called *edges*. The set of all edges is here denoted by $\mathcal{E}$.

The natural composition of curves leads to a composition of edges, $(e_2, e_1) \mapsto e_2 e_1$, if the final point of e_1 coincides with the initial point of e_2. The inverse e^{-1} of an edge e is the edge which differs from e only in the orientation.

A generalized connection $\bar{A}$ is map $\bar{A} : \mathcal{E} \to G$ such that

$$\bar{A}(e^{-1}) = \left(\bar{A}(e)\right)^{-1} \quad \text{and} \quad \bar{A}(e_2 e_1) = \bar{A}(e_2)\bar{A}(e_1), \tag{1}$$

whenever the composition of edges $e_2 e_1$ is again an edge.[2] The set of all such

---

[2]The composition of edges does not necessarily produce new edges. In general one obtains

maps $\bar{A}$ (with an appropriate topology) is the space of generalized connections $\bar{\mathcal{A}}$.

Following [6, 10], let us introduce the notion of *germ*, for edges. Consider the following equivalence relation in $\mathcal{E}$: two edges are equivalent if they start at the same point and intersect at an infinite number of points, in which case one of the edges is an analytic extension of the other. An equivalence class of edges is called a germ. Thus, a germ at a point $x \in \Sigma$ is characterized by an infinite number of Taylor coefficients (in some parametrization) at x, from which the whole family of equivalent edges can be reconstructed. The set of germs corresponding to a given point $x \in \Sigma$ is independent of x, and will be denoted by $\mathcal{K}$. The germ of e is denoted by $[e]$.

Since G is a group, the set $\mathrm{Map}[\Sigma \times \mathcal{K}, \mathrm{G}]$ of all maps $g : \Sigma \times \mathcal{K} \to G$ is a group under pointwise product, i.e. $(gg')(x, [e]) = g(x, [e])g'(x, [e])$ defines a group structure. The main object of the present paper is a subgroup of this group, defined as follows.

Definition 1   *i) For a given edge e and a point x on e, let e_x denote the subedge of e that starts at x, i.e. e_x starts at x, has the same orientation as e, and $e_x \cap e = e_x$.*

  ii) For a given $g : \Sigma \times \mathcal{K} \to G$ and an edge e, let $S(g, e)$ denote the set of points x along the edge e such that $g(x, [e_x])$ is different from the identity of the group G.

  iii) The subset $\mathcal{F} \subset \mathrm{Map}[\Sigma \times \mathcal{K}, \mathrm{G}]$ is defined as the set of all elements g such that $S(g, e)$ is a finite set for every given edge e.

It is straightforward to see that $\mathcal{F}$ is a subgroup of $\mathrm{Map}[\Sigma \times \mathcal{K}, \mathrm{G}]$, since if $S(gg', e)$ is an infinite set for some edge e, then either $S(g, e)$ or $S(g', e)$ (or both) must be infinite.

Proposition 1 *$\mathcal{F}$ is a group with respect to the pointwise product.*

The following are examples of elements of $\mathcal{F}$.

E1. g supported on a finite number of points, i.e. g is different from the identity of G only in a finite number of points of Σ. On that set, g may depend arbitrarily on the germs.

E2. g supported in an analytic line ℓ, such that g also equals the identity for every germ that defines curves on the line ℓ. Appart from that, g may vary from point to point on ℓ and depend on the germ at each point.

E3. g supported in an analytic surface S. Like in the **E2** case, g is also required to equal the identity for every germ that defines curves on S.

---

equivalence classes of piecewise analytic curves. The equivalence relation in the set of piecewise analytic curves includes also modding out by retracings, so that two curves differing in orientation are inverses of each other. The thus obtained algebraic structure is a groupoid, and generalized connections are morphisms, with values in G.

Examples **E2** and **E3** are well defined due to the following property: the number of points x in the intersection between an analytic edge e and an analytic surface S (line ℓ) such that $[e_x]$ does not define curves in S (ℓ) is finite. The same is true if the surface S (line ℓ) is of the semianalytic type considered in [11]. So, $S(g, e)$ is a finite set in such cases. (To be precise, a germ $[e]$ at a point of a surface S defines a curve in S if there is an edge e in the equivalence class $[e]$ such that $e \cap S = e$.)

A subgroup of $\mathcal{F}$ is obtained when we consider functions $g \in \mathcal{F}$ that depend not on the full information carried by the germ, but only on the tangent direction of the germ, i.e. such that, at each point, $g(x, [e]) = g(x, [e'])$ whenever germs $[e]$ and $[e']$ have the same tangent direction at the starting point x. (More precisely: representative curves c_e and $c_{e'}$, in some parametrization, have the same tangent direction at the point x.) Note that this subgroup can be defined exactly like $\mathcal{F}$, with $\mathcal{K}$ being replaced by the set S^2 of (oriented) directions in the tangent space $T_x \Sigma$ at a point x, as S^2 coincides with the quotient space of $\mathcal{K}$ by the equivalence relation that identifies germs with the same tangent direction.

Definition 2 *The subgroup of $\mathcal{F}$ of those elements g that depend only on the tangent direction of the germ will be denoted by $\mathcal{TF}$.*

Special elements of $\mathcal{TF}$ – directly related to the LQG flux variables – are determined as follows.

E4. Let S be an oriented analytic (or semianalytic) surface and g a G-valued function on S. An element $g \in \mathcal{TF}$ is obtained by declaring that g is supported on S and, on points $x \in S$: *i)* g is the identity for every germ that defines curves on S; *ii)* $g(x, [e]) = \mathrm{g}(x)$ for germs that define curves pointing upwards (with respect to the orientation of S); *iii)* $g(x, [e]) = \mathrm{g}(x)^{-1}$ for germs that define curves pointing downwards.

General elements of $\mathcal{TF}$, and among them, compositions of elements of the type **E4**, will have a more general dependence on the direction at each point.

Though the following is not the aspect we are most interested in, it is important to realize that the full set $\mathcal{F}$ can be injectively mapped into the space of generalized connections $\bar{\mathcal{A}}$. The image of $\mathcal{F}$ by this map includes all the examples of distributional connections given in [1, 6].

Proposition 2 *There is an injective map $\mathcal{F} \to \bar{\mathcal{A}}$, $g \mapsto \bar{A}_g$.*

The images $\bar{A}_g$, which are defined by their actions $\bar{A}_g(e)$ on edges, are constructed as follows (using the same procedure as for the construction of graphs adapted to surfaces, or standard graphs [6]). Given $g \in \mathcal{F}$ and an edge e, let us denote the starting and ending points of e by x and y, respectively. The finite sets $S(g, e)$ and $S(g, e) \cup \{x, y\}$ are naturally ordered,[3] following the orientation of e. One starts by decomposing e using the set $S(g, e)$, i.e. the edge e is

---

[3]It may happen that $S(g, e) \cap \{x, y\} \neq \emptyset$, in which case only one copy of x or y is kept in the set $S(g, e) \cup \{x, y\}$. On the other hand, if $S(g, e) = \emptyset$ then $\bar{A}_g(e) = \mathbf{1}$.

written as a composition of subdges $e = a_n a_{n-1} \cdots a_1$, where n is some integer, $a_k \cap e = a_k$ and the a_k's are edges starting and ending at consecutive points of the set $S(g,e) \cup \{x,y\}$, $\forall k = 1, \ldots, n$. Next, if necessary, one further decomposes the edges a_k, so that e is written as a composition of subdges $e = b_m b_{m-1} \cdots b_1$, $m \geq n$, where each edge b_k either starts or ends (not both) at a point of $S(g,e)$. In doing so, arbitrary cuts between consecutive points of $S(g,e)$ are introduced. Finally, let us replace the edges b_k that end in points of $S(g,e)$ by their inverses, and rename the set of edges so that the edge e is written as a composition of subdges

$$e = e_m^{\epsilon_m} e_{m-1}^{\epsilon_{m-1}} \cdots e_1^{\epsilon_1}, \tag{2}$$

where each edge e_k starts at a point of $S(g,e)$ and contains no other point of that set, and the symbols ϵ_k take values ± 1.

After a decomposition of the type (2) is performed, $\bar{A}_g(e)$ is simply defined by

$$\bar{A}_g(e) = [\bar{A}_g(e_m)]^{\epsilon_m} [\bar{A}_g(e_{m-1})]^{\epsilon_{m-1}} \cdots [\bar{A}_g(e_1)]^{\epsilon_1}, \tag{3}$$

$$\bar{A}_g(e_k) = g(x_k, [e_k]), \quad k = 1, \ldots, m, \tag{4}$$

where $x_k \in S(g,e)$ is the starting point of the outgoing subedge e_k.

The first thing to notice about the above construction is that it is well defined, although the decomposition (2) is not unique (due to the arbitrariness in the choice of the subedges e_k). This follows from the fact that $\bar{A}_g(e_k)$ depends only on the germ, and not on the edge starting at x_k. Secondly, one can easily check that the maps $\bar{A}_g$ satisfy the properties (1), i.e. they define elements of $\bar{\mathcal{A}}$.

We finally come to the main results of the paper.

Proposition 3 *There is a faithful representation, hereby denoted Θ, of $\mathcal{F}$ as a group of transformations in $\bar{\mathcal{A}}$. The transformations generated by the LQG flux variables belong to the subgroup $\Theta(\mathcal{TF})$.*

To construct the representation Θ, it is sufficient to give, for each $g \in \mathcal{F}$, the images $\Theta_g(\bar{A})$ of every generalized connection, which in turn are defined once $\Theta_g(\bar{A})(e)$ is known for every edge. To define $\Theta_g(\bar{A})(e)$, let us proceed as above, and assume that a decomposition of the type (2) has been performed, corresponding to a given $g \in \mathcal{F}$ and edge e. Then

$$\Theta_g(\bar{A})(e) = [\Theta_g(\bar{A})(e_m)]^{\epsilon_m} [\Theta_g(\bar{A})(e_{m-1})]^{\epsilon_{m-1}} \cdots [\Theta_g(\bar{A})(e_1)]^{\epsilon_1}, \tag{5}$$

with

$$\Theta_g(\bar{A})(e_k) = \bar{A}(e_k)\, g^{-1}(x_k, [e_k]), \quad k = 1, \ldots, m, \tag{6}$$

where $x_k \in S(g,e)$ is the starting point of the edge e_k.[4]

As above, one can check that the expressions (5, 6) are independent of the particular decomposition (2). It is also easy to convince oneself that $\Theta_g(\bar{A})$

---

[4]The appearence of g^{-1} in expression (6), instead of g, is simply related to the way in which we write the composition of edges. Also, we are only discussing the nontrivial situation $S(g,e) \neq \emptyset$, otherwise $\Theta_g(\bar{A})(e) = \bar{A}(e)$.

belongs to $\bar{\mathcal{A}}$, for every g and $\bar{A}$ (i.e. that conditions (1) are satisfied), and that the mapping $g \mapsto \Theta_g$ is injective.

It remains to see that Θ is a representation. The identity of $\mathcal{F}$ is obviously mapped to the identity transformation, so let us just consider the relation $\Theta_{gg'} = \Theta_g \circ \Theta_{g'}$. For given g and g', generalized connection $\bar{A}$ and edge e, the relation $\Theta_{gg'}(\bar{A})(e) = \Theta_g \circ \Theta_{g'}(\bar{A})(e)$ is clearly satisfied in the two cases $S(g, e) \cap S(g', e) = \emptyset$ and $S(g, e) = S(g', e)$. On the other hand, for each fixed edge e, the general case can be reduced to a combination of the above two cases, and the representation property follows.

Thus, the group $\mathcal{F}$ can be seen as a group of transformations in the space of generalized connections. Let us consider in particular elements $g \in \mathcal{TF} \subset \mathcal{F}$ of the type **E4** above, with the G-valued functions $\mathrm{g}(x)$ being generated by $\mathrm{Lie}(G)$-valued functions $f(x)$, i.e. $\mathrm{g}(x) = \exp(t f(x))$, $t \in \mathbb{R}$. The transformations Θ_g corresponding to these elements are precisely the transformations in $\bar{\mathcal{A}}$ generated by the LQG flux variables (see equations (17) in [11]).

Like the transformations generated by the flux variables, the transformations corresponding to the group $\mathcal{F}$ are continuous with respect to the natural topology of $\bar{\mathcal{A}}$, and leave the Ashtekar-Lewandowski measure [1] invariant. The proof is essentially the same as in the case of the flux variables [11, 12, 14].[5]

Proposition 4 *The Ashtekar-Lewandowski measure is $\mathcal{F}$-invariant.*

3   Conclusion

We have introduced groups of transformations in the space of generalized connections $\bar{\mathcal{A}}$ that include all the transformations generated by flux variables. This constitutes an advance in the characterization and understanding of the kinematical structure of LQG.

The group $\mathcal{F}$ is an infinite-dimensional and non-abelian subgroup of a very large product group. The latter is made of copies of the gauge group G, with one copy per each germ of analytic curves and per point in space. Elements of $\mathcal{F}$ can then be thought of as certain G-valued functions on the 'bundle of germs over the spatial manifold'. Note that those functions typically equal the identity of G at most points, with non-trivial values occurring in lower dimensional regions of space. Elements of $\mathcal{F}$ then act on $\bar{\mathcal{A}}$, with the behaviour of generalized connections typically being transformed only over lower dimensional regions.[6] This means that values $\bar{A}(e)$ can be affected only for edges intersecting those regions. A corresponding description of the group $\mathcal{TF} \subset \mathcal{F}$ and its action can be made, with the difference that functions and transformations no longer depend on the full information carried by germs, but only on the direction.

An important property of the group $\mathcal{F}$ is that its action leaves the Ashtekar-Lewandowski measure invariant. It follows that the standard LQG representa-

---

[5]For each fixed g, every graph in Σ can be split and generators chosen such that the transformation Θ_g is a right translation in the space G^N labelled by the graph.

[6]The action Θ is most simple in the abelian case $G = U(1)$. In that situation $\bar{\mathcal{A}}$ is itself a group, $\mathcal{F}$ is a subgroup of $\bar{\mathcal{A}}$ and Θ is the natural action of $\mathcal{F} \subset \bar{\mathcal{A}}$ on $\bar{\mathcal{A}}$.

tion of the holonomy-flux algebra carries an unitary representation of the group of transformations $\Theta(\mathcal{F})$, which extends the set of unitary operators corresponding to the quantization of the transformations generated by the flux variables.

Even if the exact physical meaning of this unitary representation of the full group $\Theta(\mathcal{F})$ is not yet clear, it is our opinion that the group of transformations $\Theta(\mathcal{F})$ deserves attention and further study, as happened e.g. with another extended group of symmetries of the Ashtekar-Lewandowski measure, namely the group of automorphisms of the groupoid of paths [8, 9]. In particular, it is tempting to use the criteria of unitary implementability of $\mathcal{F}$, or at least of its subgroup $\mathcal{TF}$, to explore further the representation theory of the holonomy flux-algebra, hopefully beyond the beautiful results obtained by Lewandowski, Okolow, Sahlmann, Thiemann and Fleischhack [10, 11, 12, 14]. In this respect, note that it already follows from those results (see e.g. Lemma 6.2 in [14]) that the Ashtekar-Lewandowski measure is the only $\mathcal{TF}$-invariant measure in $\bar{\mathcal{A}}$.[7]

Acknowledgements

References

[1] A. Ashtekar and J. Lewandowski, Representation Theory of Analytic Holonomy $C^{\star}$ Algebras, in *Knots and Quantum Gravity*, ed J. Baez (Oxford U. Press, Oxford, 1994).

[2] A. Ashtekar A and J. Lewandowski, Differential geometry on the space of connections via graphs and projective limits, *J. Geom. Phys.* **17** (1995) 191-230.

[3] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão and T. Thiemann, Quantization of diffeomorphism invariant theories of connections with local degrees of freedom, *J. Math. Phys.* **36** (1995) 6456-6493.

[4] J. Baez, Generalized measures in gauge theory, *Lett. Math. Phys.* **31** (1994) 213-224.

J. Baez, Spin networks in gauge theories, *Adv. Math.* **117** (1996) 253-272.

[5] A. Ashtekar and J. Lewandowski, Background independent quantum gravity: a status report, *Class. Quantum Grav.* **21** (2004) R53.

---

[7]Measures μ that are invariant under a transformation T provide a unitary implementation of T in the Hilbert space defined by μ. The same is true for quasi-invariant measures, i.e. such that μ and the push-forward measure $T_*\mu$ are mutually absolutely continuous.

[6] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge U. Press, Cambridge, 2007).

[7] J. M. Velhinho, A groupoid approach to spaces of generalized connections, *J. Geom. Phys.* **41** (2002) 166-180.

[8] J. M. Velhinho, On the Structure of the Space of Generalized Connections, *Int. J. Geom. Meth. Mod. Phys.* **1** (2004) 311-334.

J. M. Velhinho, Functorial aspects of the space of generalized connections, *Mod. Phys. Lett.* A **20** (2005) 1299-1303.

[9] B. Bahr and T. Thiemann, Automorphisms in loop quantum gravity, arXiv:0711.0373.

[10] H. Sahlmann and T. Thiemann, On the superselection theory of the Weyl algebra for diffeomorphism invariant quantum gauge theories, gr-qc/0302090.

[11] J. Lewandowski, A. Okolow , H. Sahlmann and T. Thiemann, Uniqueness of diffeomorphism invariant states on holonomy-flux algebras, *Commun. Math. Phys.* **267** (2006) 703-733.

[12] C. Fleischhack, Representations of the Weyl algebra in quantum geometry, math-ph/0407006.

[13] C. Fleischhack, Construction of generalized connections, math-ph/0601005.

[14] A. Okolow and J. Lewandowski, Diffeomorphism covariant representations of the holonomy flux $\star$-algebra, *Class. Quantum Grav.* **20** (2003) 3543-3568.